INTEGRAL SOLUTION OF THE TERNARY CUBIC EQUATION

$$5(x^2+xy^2)-9xy+x+y+1=28z^3$$

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ABSTRACT

The Ternary cubic equation $5(x^2+y^2)-9xy+x+y+1=28z^3$ is considered for determining its non-zero distinct integral solutions. Employing the linear transformation $x=u+v, y=u-v,$ and employing the method of factorization in complex conjugates, different patterns of integral solution to the ternary cubic equation under consideration are obtained in each pattern, interesting relations among the solutions and some special polygonal numbers like pyramidal, central pyramidal numbers are exhibited.

KEY WORDS: Ternary Cubic, Integral Solutions, Polygonal Number, Pyramidal number.

1. INTRODUCTION

Diophantine equations are numerously rich because of its variety. The determination of integral solutions for Cubic (homogeneous or non-homogeneous) diophantineequations with three variables has been an interest to mathematicians. Since antiquity as can be seen from [1-3]. In this context one may refer [4-24]. In this communication, the non-homogeneous ternary cubic diophantine equation represented by $5(x^2+y^2)-9\mathrm{xy}+\mathrm{x}+\mathrm{y}+1=28z^3$ is considered for its non-zero distinct integral solution. A few interesting relations between special polygonal numbers and Pyramidal numbers are exhibited.

NOTATIONS USED

1.	Pr_n	-	Pronic number of rank n
2.	S_n	-	Star number of rank n
3.	So_n	-	Stella octangularnumber of rank n
4.	OH_n	-	Octangular number of rank n
5.	Gno_n	-	Gnomic number of rank n
6.	Obl_n	-	Oblong number of rank n
7.	PP_n	-	Pentagonal Pyramidal number of rank n
8.	CP_n^6	-	Centred hexagonal pyramidal number
9.	CP_n^3	-	Centred hexagonal pyramidal number
10.	O_n	-	Octahedral number of rank n
11.	$4DF_n$	-	Four dimensional figurate number
12.	$T_{m,n}$	-	Polygonal number of rank n with size m

2. Methods of Analysis

The Ternary Cubic Diophantine equation under consideration is

$$5(x^2+y^2)-9xy+x+y+1=28z^3 \tag{1}$$

Introduction of the transformations $\mathrm{x}=\mathrm{u}+\mathrm{v}, \mathrm{y}=\mathrm{u}-\mathrm{v}$ (2)
In (1) leads to

$$(u+1)^2+19v^2=28z^3 \tag{3}$$

Equation (3) is solved through different methods and thus,we obtain different patterns of Solutions to (1).

2.1 Method 1

Take $z=a^2+19b^2$ (4)

Write$28=(3+i\sqrt{19})(3-i\sqrt{19})$ (5)

Using (4) and (5) in (3) and employing the method of factorization, define

$$(u+1)+i\sqrt{19}v=\left(3+i\sqrt{19}\right)(a+i\sqrt{19}b)^3 \tag{6}$$

Equating real and imaginary parts of (6) on both sides, we get

$u+1=3a^3-57a^2b-171ab^2+361b^3$
$v=a^3+9a^2b-57ab^2-57b^3$

Substituting the values of u,v in (2), the non-zero distinct integralsolutions to (1) are given by

$x=4a^3-48a^2b-228ab^2+304b^3-1$
$y=2a^3-66a^2b-114ab^2+418b^3-1$
$z=a^2-19b^2$

PROPERTIES

1. $x(a,1)-y(a,1)-So_n-2t_{4,3a}\equiv 114 \pmod{113}$
2. $x(a,1)-4CP_{6,a}+48t_{4,a}-303\equiv 0 \pmod{228}$
3. $y(a,1)-x(a,1)+4CPa^3+36t_{2,a}\equiv 114 \pmod{134}$
4. $x(a,1)+z(a,1)-4Cub_n-S_n+53t_{4,a}\equiv 93 \pmod{16}$
5. $z(a,a(a+1))-228FN_n^4+38CP_n^6-39t_{4,a}=0$
6. $x(a,a+1)-16So_n-408Pr_a\equiv 303 \pmod{292}$
7. $z(a,a+1)-5t_{4,2a}\equiv 19 \pmod{38}$

Note:
Re-writing (5) as

$$28=\left(-3+i\sqrt{19}\right)\left(-3-i\sqrt{19}\right)$$

And proceeding as in method 1, the non-zero distinct integralsolutions to (1) are given by

$x=-2a^3-66a^2b+114ab^2+418b^3-1$
$y=-4a^3-48a^2b+228ab^2+304b^3-1$
$z=a^2+19b^2$

2.2 Method 2

Equation (3) can be written as

$$(u+1)^2+19=28z^3*1$$

Write '1' as

$$1=\frac{(9+i\sqrt{19})(9-i\sqrt{19})}{10^2}$$

Define

$$\left(u+1+i\sqrt{19}\;v\right)=\frac{(3+i\sqrt{19})(9+i\sqrt{19}\;b)^3\;(9+i\sqrt{19})}{10} \tag{7}$$

Equating real and imaginary parts, we have

$$u+1=\frac{1}{10}\ [8a^3+4332b^3-456ab^2-684a^2\mathrm{b}] \quad (8)$$

$$v=\frac{1}{10}\ [12a^3-152b^3-684ab^2+24a^2\mathrm{b}] \quad (9)$$

Since our aim is to find integer solution, assuming$a=10A$and $b=10B$ in (4), (8) and (9) and substituting the values of u,v in (2), we obtain the distinct non-zero integral solutions to (1) as

$x=10^2\ [20A^3+4180B^3-1140AB^2-660A^2\mathrm{B}]-1$
$y=10^2\ [-4A^3+4484B^3-228AB^2-708A^2\mathrm{B}]-1$
$z=10^2\ [A^2+19B^2]$

PROPERTIES

1. $z(A,A(A+1))-\ 22800FN_A^4+\ 3800CP_A^6\ +39T_{4,10A}\ =0$
2. $y\ (A,1)+\ 600\ O_A\ +\ 70800\ (0b1)_A\ \equiv 448399\ (mod\ 48200)$
3. $z(A,A^2)-\ 91200DF_A\ -\ 20t_{4,10A}\ =0$
4. $z\ (A,A+1)+\ 2500Pr_A\ +\ T_{1002,A}\ \equiv 1900$ (mod 801)
5. $x\ (A,1))+\ 2000CP_n^6+\ 13200T_{12,A}\ \equiv 417999\ (mod\ 166800)$
6. $x\ (A,1)-\ 5y(A,1)-\ 2880\ T_{4,10A}\ =1823996$

2.3 Method 3

Instead of (5), 28 can be written as

$$28=\frac{(6+2i\ \sqrt{19})(6-2i\sqrt{19})}{2^2} \quad (10)$$

Proceeding as in method 1 and performing some algebra, we obtain the distinct non-zero integral solutions to (1) as

$x=2^2\ [8A^3+608B^3-96A^2B-456AB^2]-1$
$y=2^2\ [4A^3+836B^3-132A^2B-228AB^2]-1$
$z=2^2\ [A^2+19B^2]$

PROPERTIES

1. $x\ (A,1)+\ 32\ CP_A^6+\ 192\ T_{6,A}\ \equiv 2431\ (mod\ 1632)$
2. $x\ (A,A+1)+\ 896\ SO_A\ +\ 1216\ Pr_A+\ 768P_A^5\ \equiv 2431\ (mod\ 3360)$
3. $y\ (A,1)-\ 4Az(A,1)+\ 33T_{4,4A}\ \equiv 3343\ (mod\ 1216)$
4. $x\ (A,1)-\ 2y(A,1)-\ 672T_{4,A}\ =9119$
5. $z\ (A,A(A+1))-\ 912FN_A^4+\ 152CP_A^6+39T_{4,2A}\ =0$
6. $z\ (A,A+1)-\ 40\ T_{6,A}\ \equiv 76\ (mod\ 192)$

Note:

Re-write (7) as

$$(u+1)+i\sqrt{19}v=(-3+i\sqrt{19})(a+i\sqrt{19}b)^3\left(\frac{9+i\sqrt{19}}{10}\right)$$

and proceeding as in method 2, the non-zero distinct integralsolution to (1) are given by

$x=2^2\ [-4A^3+836B^3-132A^2B+228AB^2]-1$
$y=2^2\ [-8A^3+608B^3-96A^2B+456AB^2]-1$
$z=2^2\ [A^2+19B^2]$

2.4 Method 4

Equation (7) can be written as

$$(u+1)+i\ \sqrt{19}v=\frac{(6+2i\ \sqrt{19})}{2}\ (a+i\ \sqrt{19}b)^3\frac{(9+i\ \sqrt{19})}{10} \quad (11)$$

Equating real and imaginary Parts on both sides

$$u+1=\frac{1}{20}\ [16a^3+8664b^3-1368a^2\mathrm{b}-912ab^2] \quad (12)$$

$$v=\frac{1}{20}\ [24a^3-304b^3+48a^2\mathrm{b}-1368ab^2] \quad (13)$$

Since our aim is to find integer solutions, assuming a=20A, b=20B in (4), (12) and (13) and substituting the values of u,v in (2), we obtain the distinct non-zero inegral solutions to (1) as

$x=20^2\ [40A^3+8360B^3-1320A^2B-2280AB^2]-1$
$y=20^2\ [-8A^3+8968B^3-1416A^2B+456AB^2]-1$
$z=20^2\ [A^2+19B^2]$

PROPERTIES

1. $x\ (A,1)-\ 40Az\ (A,1)+\ 105600\ T_{12,A}\ \equiv 3343999\ (mod\ 1321600)$
2. $z\ (A,A+1)+\ 4000\ T_{6,A}\ \equiv 7600\ (mod\ 19200)$
3. $y(A,1)-1600\ S0_A\ +\ 1416T_{4,20A}\ \equiv 3587199\ (mod\ 180800)$
4. $x(A,1)-\ y(A,1)-\ 28800\ OH_A\ -9600\ T_{10,A}\ \equiv 243200\ (mod\ 1075200)$
5. $x\ (A,A+1)-3840000\ PP_A\ -3600\ T_{4,40A}\ -4560000\ Gno_A\ =37999999$

Conclusion

In this paper, we have made an attempt to obtain a complete set of non-trivial distinct integral solutions for the non-homogeneous ternary cubic equation. To conclude, one may search for other choices of solutions to the considered cubic equation and further cubic equations with multivariables.

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